PHYS1110D – Engineering Physics: Mechanics and Thermodynamics

Supplementary Material for Week 3: The Principle of Least Action; Euler-Lagrange Equation

*The final problem of Week 3 (the Principle of Least Action) is an “unauthorized” pedagogical experiment in the tutorial sessions of PHYS1110D only. As far as we know, nobody has tried to teach this principle in an introductory physics course for engineering students. Our approach is also different from the traditional way of teaching it, hoping that we can make the principle easier to learn for students with little experience in calculus.*

*Some students followed the TA with great enthusiasm, feeling that the principle was really interesting. But we also know that there are many of you who had great trouble when learning it, and we feel sorry about that. For these students, we really want to convince you that the Principle of Least Action is a very exciting law of nature. You can follow the steps in the uploaded solutions to see if you can understand it better.*

In our teaching, many students find that the definition of Lagrangian is not well motivated. Historically, Newton’s Second Law was discovered earlier. But the differential equation that describes Newton’s Law also lacks some “beauty”: why does nature choose such a relation? At that time, people have discovered **the Principle of Least Time (Fermat’s Principle)** in geometric optics:

*For given starting and ending position, light will choose the path that takes the shortest time to travel.*

Inspired by this, physicists worked hard to find some similar extremum theory about the motion of ordinary objects. The result of their work is **the Principle of Least Action**:

*For given starting and ending position* ***and time****, objects will choose the path that has the least action.*

In this principle, the somewhat artificial “action” replaces time in Fermat’s Principle. But this action is constructed with some hypothesis:

* It should be something that takes the *whole path* into account.
* In classical mechanics, once knowing the *velocity* and the *position*, the motion of the particle afterwards is completely determined. Thus, the action should involve position and velocity only, excluding acceleration and any other higher-order time-derivatives of the position.

Then physicist suggest that the action is an integral of the form

The integrand is called the **Lagrangian**, a function of the position, the velocity and the time.

Interestingly, it turns out that the Lagrangian describing free particles (which is also in agreement with special relativity) is again the time! It is

is the speed of light. The quantity

is called the **proper time**, which is the time recorded by a clock *moving together with the object*.

Now let’s return to the classical world and not worry about the relativistic effects. Our goal is to determine the condition of least action, with some abstract Lagrangian to be specified later.

By definition of integral, we can write the action as

The condition for least action is

Here we used the chain rule for multi-variable functions:

We notice that

Taking the limit (), we finally obtain the desired result

This is called the **Euler-Lagrange Equation**. Comparing with Newton’s Second Law, we see that in classical mechanics, the Lagrangian should satisfy

Therefore, the classical Lagrangian was chosen to be

This is **for classical mechanics only**. It is the correction of the Lagrangian that lead us to special relativity, electrodynamics, gravitation and so much more. Note that in the classical Lagrangian, the time only appears in and . So, people say that the Lagrangian does not **explicitly** depend on time.